DRP Presentation

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Definition of φ

$$\varphi : \mathbb{N} \to \mathbb{N}$$

$$n = p_1^{a_1} p_2^{a_2} .. p_k^{a_k} \longmapsto (p_1 + a_1)(p_2 + a_2)..(p_k + a_k)$$

It is clear that φ is multiplicative.
I.e. if $m, n \in \mathbb{N}$ and $\gcd(m, n) = 1$ then $\varphi(mn) = \varphi(m)\varphi(n)$.

Examples

$n = 12,$

$$\varphi(12) = \varphi(2^2 \cdot 3) = \varphi(2^2)\varphi(3^1) = (2+2)\cdot(3+1)$$

$$= (4)\cdot(4) = 16$$

$n = 360,$

$$\varphi(360) = \varphi(2^3 \cdot 3^2 \cdot 5^1) = \varphi(2^3)\varphi(3^2)\varphi(5^1) = (2+3)\cdot(3+2)\cdot(5+1)$$

$$= 5 \cdot 5 \cdot 6 = 150$$

Known Cycles

(a) 1
(b) 4
(c) 90
(d) 120
(e) 20,24
(f) 6,12,16

Behavior of φ on p^a

For what p^a will we find, $\varphi(p^a) > p^a$, $\varphi(p^a) = p^a$ or $\varphi(p^a) < p^a$?
Perhaps this can can yield understanding of φ on $n = p_1^{a_1} \cdot p_2^{a_2} .. p_k^{a_k}$
Two cases:

1. $p = 2$
2. $p > 2$

$p = 2$

Now consider the cases of a

When $a = 1$ we see $2 = 2^1 < \varphi(2^1) = 3$.
When $a = 2$ is is easy to see that $2^2 = \varphi(2^2) = 4$.
When $a = 3$, we have $8 = 2^3 > \varphi(2^3) = 5$.

It is clear that $\frac{p^{a+1}}{\varphi(p^{a+1})} > \frac{p^a}{\varphi(p^a)}$ and so for all $a > 2$ we have $2^a > \varphi(2^a)$

$p > 2$

When $a = 1$ it is clear that $p^1 = p < p + 1 = \varphi(p^1)$.
I.e. for all primes p, $p^1 < \varphi(p^1)$.

When $a > 1$ we have $\frac{p^a}{\varphi(p^a)} > \frac{2^a}{\varphi(2^a)} \geq 1$ and so $p^a > \varphi(p^a)$.

Cycles of Length 1/Fixed points

We may ask about the behavior of φ as we apply it recursively on its output.

We now attempt to show that there is a finite number of fixed points (integers n s.t. $\varphi(n) = n$)

We will now try to identify all integers n s.t. $\varphi(n) = n$.

Useful convention

It will be convenient for us to write any candidate n in the form

$$n = 2^t \prod p_i^{a_i} \prod q_i^{b_i}$$

where $p_i^{a_i}$ are the components of n not divisible by 2 that φ shrinks ($a_i > 1$) and $q_i^{b_i}$ are the components of n not divisible by 2 that φ grows ($b_i = 1$)

Since every $b_i = 1$,

$$n = 2^t \prod p_i^{a_i} \prod q_i$$

Case $t = 0$

We need to find a solution to,

$$\prod p_i^{a_i} \prod q_i = \prod (p_i + a_i) \prod (q_i + 1)$$

Notice that if $n = 1$ we have a trivial solution as the above.
If $n \neq 1$ is a solution then both $\prod p_i^{a_i}$ and $\prod q_i^{b_i}$ are not the empty product.
Notice that $2 \mid (q_1 + 1)$ yet $2 \nmid n$ and so we cannot have a non-trivial solution where $t = 0$.
Solutions where $t = 0$: $n = 1$.

Cases $t = 1, 2, 3$

Solutions where $t = 1$**:** $n = 90$**.**
Solutions where $t = 2$**:** $n = 4$**.**
Solutions where $t = 3$**:** $n = 120$**.**

Case $t > 3$

We take $L_t := \frac{2^t}{2+t}$
Notice for $t > 3$ we have,

$$\frac{8}{3} \leq L_t$$

We let ρ_i denote the i^{th} odd prime
Next we take $S_t := \prod \frac{\rho_i+1}{\rho_i}$ we get
Notice for $t = 4$,

$$S_t = \frac{4 \cdot 6 \cdot 8 \cdot 12}{3 \cdot 5 \cdot 7 \cdot 11} < L_t = \frac{8}{3}$$

We now say that for $t = k$ we have $S_t < L_t$
We take $l_t := \frac{2(2+t)}{3+t}$ and
take $s_t := \frac{12}{11}$
Notice,

$$L_{t+1} = L_t l_t$$

and,

$$S_{t+1} \leq S_t r_t$$

Notice,

$$\frac{12}{11} = r_t < \frac{12}{7} \leq l_t$$

for all $t > 3$ Since $S_t < L_t$ and $s_t < l_t$

$$S_{t+1} \leq S_t s_t < L_t l_t = L_{t+1}$$

We now have shown induction,

Since we have a base case with $t = 4$ we now see that $S_t < L_t$ for all $t > 3$
We assume that there is a solution where $t > 3$
We let n_t be such a solution with specific $t > 3$
We write n_t in the form we have seen earlier,

$$n_t = 2^t \prod p_i^{a_i} \prod q_i$$

and take $R_t := \prod \frac{q_i+1}{q_i}$

Notice we have,

$$2^t \prod p_i^{a_i} \prod q_i = (2+t) \prod (p_i + a_i) \prod (q_i + 1)$$

and so,

$$L_t = \frac{\prod (p_i + a_i) \prod (q_i + 1)}{\prod p_i^{a_i} \prod q_i} \leq \prod \frac{q_i + 1}{q_i} \leq S_t < L_t$$

this is a contradiction,
And so we see that there cannot be a solution where $t > 3$
Soltuions where $t > 3$: N/A

Complete list of fixed points

$$n = 1, 4, 90, 120$$